Activate Prior Knowledge

Ex. 1) $(2x + 3)(4x + 5)$

$2x(4x + 5) + 3(4x + 5)$

$8x^2 + 10x + 12x + 15$

$8x^2 + 22x + 15$

- Notice the colors are helpful, but adding in the arrows can add to students understanding of how to split the first binomial.
- It's important to have students see that the operation within the first binomial must travel with the second term when setting up the distributive property.
- When making the marking of your choice under the like terms as in the 3rd line, be sure to extend this marking to include the operation before the first like term.
- Make a point to stress the final middle term came from combining the two middle terms in the 3rd line.

Ex. 2) $(3x – 4)(2x + 3)$

$3x(2x + 3) – 4(2x + 3)$

$6x^2 + 9x – 8x – 12$

$6x^2 + x – 12$

- This example illustrates how to distribute a negative value over a binomial. This is another reason why it's important to stress taking the operation of the second term in the first binomial when setting up the distributive property.
- When distributing a negative through, students often forget their integer operation rules and forget to distribute the sign along with the number to the second term in the parenthesis.

How to Factor Using the Reverse Distributive Method

The premise of this method is to create a factor by grouping scenario so that students can write the trinomial in an equivalent product of binomials. To do this we need to know how to split the middle term into to two terms as in the previous examples 4th and 3rd line respectively. There are many ways to make the value of 22x and 1x as in the examples above, but to find the right combination of terms we need to utilize a small table and the standard form of a quadratic trinomial.

Standard Form $ax^2 + bx + c$

Table

Two #'s that multiply to give "ac"	Two #'s that add to give "b"

Let's take the first example slowly.

Ex 3.) $x^2 + 10x + 21$

Matching to the standard form
$a = 1 \quad b = 10 \quad c = 21$

Creating the table

Mult. **ac** **1 · 21 = 21**	**Add** **b** **10**

I feel it is best to start with the product side of the table. This will come into play most when the product of "ac" is negative.

Mult. **ac** **1 · 21 = 21**	**Add** **b** **10**
1 · 21	1 + 21
-1 · -21	-1 + -21
3 · 7	3 + 7
-3 · -7	-3 + -7

Mult. to give 21 → (multiplication column)

Adds to give 10 → (addition column)

Now that we have found the values that satisfy both of these conditions, we will use these two values to rewrite the coefficient of the middle term in the original trinomial.

$$x^2 + 10x + 21$$

$$x^2 + 7x + 3x + 21$$

The order of these two middle terms does not matter.

Now simply factor by grouping by factoring out the Greatest Common Factor (GCF) from the first two terms and then again from the last two terms.

$$x^2 + 7x + 3x + 21$$

$$x(x + 7) + 3(x + 7)$$

Notice that the two terms have a common factor of (x + 7). We can use color to highlight this and make reference to when we found the product of two binomials.

$$x(x + 7) + 3(x + 7)$$

Factoring out the GCF of (x + 7) would give

$$(x + 7)(x + 3)$$

or

$$(x + 3)(x + 7)$$

Commutative Property

$$x^2 + 10x + 21 = (x + 3)(x + 7)$$

A few things to note

- The purpose of the table is to find two values that satisfy both conditions.
- Once these two values have been determined, rewrite the middle term of the original trinomial, and then factor by grouping.
- If need you can add the following in your initial examples to explain how we are splitting the middle term properly:

$x^2 + 10x + 21$

$x^2 + (7 + 3)x + 21$

$x + 7x + 3x + 21$

- If two values cannot be determined using the table, then the trinomial is "Prime"
- This lesson focuses on factoring trinomials. It is to be said that the GCF of the trinomial must be factored out prior to starting this method.
- This method will work on ALL trinomials that are factorable. This includes trinomials that have $a > 1$, as well as the special trinomials and binomials that can be written in the standard form of a quadratic trinomial.
- This method will also work when the exponential value of the variable is a multiple of 2
- An additional variable may be included so long as it matches this example

$$ax^2 + bxy + cy^2$$

Ex. 4) $2x^2 - 15x + 7$

$a = 2$ $b = -15$ $c = 7$

Looking for two values that multiply to give 14 and add to give -15

ac = 2 · 7 = **14**	b = **-15**
1 · 14	1 + 14
-1 · -14	-1 + -14
2 · 7	2 + 7
-2 · -7	-2 + -7

Mult. to give 14 → (left column)

Adds to give -15 → (right column)

Using the values of -1 and -14, rewrite the middle coefficient of -15

$2x^2 - 15x + 7$

$2x^2 - 1x - 14x + 7$

or

$2x^2 - x - 14x + 7$

Factoring out the GCF among the first two terms and the second two terms

$$x(2x - 1) - 7(2x - 1)$$

$$(x - 7)(2x - 1)$$

$$2x^2 - 15x + 7 = (x - 7)(2x - 1)$$

A few things to note

- It makes the process easier if you have students factor out the GCF in such a way as to leave a positive leading coefficient within the second binomial. If we had not done so on this problem we would have had an extra step of factoring out a -1 before we had a common binomial among the two terms.

$$2x^2 - x - 14x + 7$$

$$x(2x - 1) + 7(-2x + 1)$$

  Students may get confused at this point if they do not adhere to this recommendation.

- When providing examples for students early in the lesson, it's important to model all of the possibilities within the table and discuss common integer mistakes. This could be an excellent class discussion on error analysis.

Ex. 5) $3x^2 - 2x - 5$

a = 3 b = -2 c = -5

Looking for two values that multiply to give -15 and add to give -2

ac = 3 · -5 = **-15**	b = **-2**
-1 · 15	-1 + 15
1 · -15	1 + -15
-3 · 5	-3 + 5
3 · -5	3 + -5

Be careful! This equals +2. (for -3 + 5)

Mult. to give -15 (→ 3 · -5)

Adds to give -2 (→ 3 + -5)

Using the values of 3 and -5, rewrite the middle coefficient of -2

$$3x^2 - 2x - 5$$

$$3x^2 + 3x - 5x - 5$$

Factoring out the GCF among the first two terms and the second two terms

$$3x(x + 1) - 5(x + 1)$$

$$(3x - 5)(x + 1)$$

$$3x^2 - 2x - 5 = (3x - 5)(x + 1)$$

A few things to note

- It's important to engage students in a whole class discussion as to how to arrive at the correct combination of integer values that satisfy both conditions within the table. Here are a few questions that you can ask students:
    - "Under what conditions can you have two numbers that multiply to give you a positive value, but then add to give you a negative value?"
        - *If you have two negative values, under multiplication the result will be a positive, but under addition the result will be negative.*
    - "Under what condition can you have two numbers that multiply to give you a positive value, but then add to give you a positive value?
        - *If you have two positive values, under both multiplication and addition the result will be positive.*
    - "Under what condition can you have two numbers that multiply to give you a negative value, but then add to give you a positive value?"
        - *If you have two values being multiplied and the result is negative, then the two values were of different signs. To add these two values, the larger value must be positive in order to obtain a positive sum.*
    - "Under what condition can you have two numbers that multiply to give you a negative value, but then add to give you a negative value?"
        - *If you have two values being multiplied and the result is negative, then the two values were of different signs. To result in a negative value after adding the two values, the larger of the two must be negative.*
- For a counter example, it would be beneficial to try to factor a trinomial using the wrong integer combination of values. This would afford students the experience of determining what this error looks like and how to correct their mistake. Picking the wrong combination of integer values is a very common mistake student make in this unit.

Ex. 6) $x^2 - 10x + 25$

$a = 1 \quad b = -10 \quad c = 25$

Looking for two values that multiply to give 25 and add to give -10

$ac = 1 \cdot 25 =$ **25**	$b =$ **-10**
$1 \cdot 25$	$1 + 25$
$-1 \cdot -25$	$-1 + -25$
$5 \cdot 5$	$5 + 5$
$-5 \cdot -5$	$-5 + -5$

Mult. to give 25 → $-5 \cdot -5$

Adds to give -10 → $-5 + -5$

Using the values of -5 and -5, rewrite the middle coefficient of -10

$$x^2 - 10x + 25$$

$$x^2 - 5x - 5x + 25$$

Factoring out the GCF among the first two terms and the second two terms

$$x(x - 5) - 5(x - 5)$$

$$(x - 5)(x - 5)$$

$$(x - 5)^2$$

$$x^2 - 10x + 25 = (x - 5)^2$$

Let's now look at a special case where the middle term is omitted in the polynomial.

Ex. 7) $x^2 - 36$

This can also be written as **$x^2 + 0x - 36$** to determine the a, b and c values.

$a = 1 \quad b = 0 \quad c = -36$

Pose the questions

- "How can we multiply two numbers and get a negative value, and then add those two numbers and get zero?"

- "How can we get zero when we add to values?"

What becomes very evident is that the two values have to be opposites.

$ac = 1 \cdot -36 = \mathbf{-36}$	$b = \mathbf{0}$
$-6 \cdot 6$	$-6 + 6$
$6 \cdot -6$	$6 + -6$

Looking for two values that multiply to give -36 and add to give 0

Using the values of -6 and 6, rewrite the coefficient of the middle term. Illustrate how to factor this polynomial both ways as shown below, so that students can analyze the effect of how they rewrite the middle term.

Method A

$x^2 + 0x - 36$

$x^2 - 6x + 6x - 36$

$x(x - 6) + 6(x - 6)$

$(x + 6)(x - 6)$

Method B

$x^2 + 0x - 36$

$x^2 + 6x - 6x - 36$

$x(x + 6) - 6(x + 6)$

$(x - 6)(x + 6)$

By way of the Commutative Property these two statements are equivalent

$$x^2 - 36 = (x + 6)(x - 6)$$

or

$$x^2 - 36 = (x - 6)(x + 6)$$

A few things to note

- Illustrating both versions of factoring reinforces the Commutative Property for students, as well as provides them experience in planning their method of factoring. Some students may prefer not to factor out a negative from the latter two terms and therefore begin factoring as shown in Method A.

- In the beginning it's important for students to rewrite their binomial as a trinomial with the middle term's coefficient equal to zero. This will allow factoring to follow the same structure as all other factoring situations, and therefore, be more comfortable for students while learning this method. Students will quickly catch on that these are special situations and are quite easy to factor.
- Most importantly, using the table and factoring by grouping prevents students from remembering all the so called "Special Cases". So often this unit is so difficult for students because they are often asked to remember different methods of factoring depending on the type of trinomial they were given. This method throws all of that early memorization out the window. This is not to say that students need not look at factoring patterns, but rather they need not be inundated with all of that in the beginning of the unit.

Ex. 8) $36x^4 - 16x^2$

First, the GCF needs to be factor from the polynomial.

$4x^2(9x^2 - 4)$

The GCF must remain as a factor of this expression. Students often want to dispose of this value, but it is essential in maintaining equivalency once the polynomial has been factored completely. Now rewrite the binomial portion as a trinomial with the middle term's coefficient equal to zero.

$4x^2(9x^2 + 0x - 4)$

Continue factoring the trinomial **$9x^2 + 0x - 4$.**

$a = 9$ $b = 0$ $c = -4$

$ac = 9 \cdot -4 =$ **-36**	$b =$ **0**
$-6 \cdot 6$	$-6 + 6$
$6 \cdot -6$	$6 + -6$

Looking for two values that multiply to give -36 and add to give 0

Using the values of -6 and 6, rewrite the middle coefficient of 0.

$$4x^2(9x^2 + 0x - 4)$$

$$4x^2(9x^2 - 6x + 6x - 4)$$

Change the parentheses to brackets for esthetic reasons.

$$4x^2[9x^2 - 6x + 6x - 4]$$

Factor the GCF among the first two terms and the second two terms.

$$4x^2[3x(3x - 2) + 2(3x - 2)]$$

$$4x^2[(3x + 2)(3x - 2)]$$

Removing brackets and writing the completely factored form

$$36x^4 - 16x^2 = 4x^2(3x + 2)(3x - 2)$$

A few things to note

- If the GCF is not initially factored out, all is not lost. The expression can still be factored using this method; however, the values will be much larger and there will be a need to factor each of the binomials before the expression is in complete factored form.

$$\begin{aligned} 36x^4 - 16x^2 &= (6x^2 + 4x) \cdot (6x^2 - 4x) \\ &= 2x(3x + 2) \cdot 2x(3x - 2) \\ &= 2x \cdot 2x \cdot (3x + 2) \cdot (3x - 2) \\ &= 4x^2(3x + 2)(3x - 2) \end{aligned}$$